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An analytical solution of the Navier–Stokes equation for internal flows

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Abstract

This paper derives a solution to the Navier–Stokes equation by considering vorticity generated at system boundaries. The result is an explicit expression for the velocity. The Navier–Stokes equation is reformulated as a divergence and integrated, giving a tensor equation that splits into a symmetric and a skew-symmetric part. One equation gives an algebraic system of quadratic equations involving velocity components. A system of nonlinear partial differential equations is reduced to algebra. The velocity is then explicitly calculated and shown to depend on boundary conditions only. This removes the need to solve the Navier–Stokes equation by a 3D numerical computation, replacing it by computation of 2D surface integrals over the boundary.

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A major obstacle for detailed calculations of fluid flows is the time consuming 3D calculations characteristic of solving differential equations numerically, or using integral methods involving the calculation of entire 3D domain integrals. For the latter approach, various methods have been presented that attempt to reduce the computational task. In the boundary element method one applies potential theory to lower the dimension of the problem by one order. However, this property is lost if vorticity is considered. An extended version of the method [1] transforms local boundary conditions into global conditions integrated over the boundary. The velocity is divided into a potential and a rotational part. This leads to a system of coupled boundary integral equations to be solved iteratively. The property of the reduced dimension is preserved. Other methods, commonly applied to vibration analysis, also transforming domain integrals into boundary integrals include the dual reciprocity method [2], and the particular integrals technique [3]. The application of these latter methods [2, 3] has been discussed in [4, 5].

Laboratory experiments indicate the significant role played by the no-slip boundary condition in the formation of vortex filaments affecting the flow evolution. Recently [6], detailed studies of vortex filaments in 2D turbulent flows reveal that their influence may have dramatic effects on the flow, not only in the vicinity of the wall but extending over the full flow domain.

It is a common assumption that to describe quantitatively turbulent flows in the presence of solid boundaries, it is necessary to have a quantitative model of the vorticity creation rate [7]. Based on this assumption, the boundary vorticity dynamics model has developed [8], describing the vorticity creation from solid boundaries and the reaction of the created vorticity to the boundaries. In this model, the velocity field is decomposed into two parts. One part, derived from a scalar potential, describes the longitudinal compression. The other part, derived from a vector potential, describes the transverse shearing process.

None of the methods or models listed in the previous paragraphs yields a closed expression for the velocity in the flow in terms of conditions at the system boundary. Such an expression is presented in this work. We do not *a priori* assume any explicit model of vorticity formation, or that the velocity field may be split into a potential and a rotational part.

From a strictly mathematical point of view, it may not be straightforward how to solve a set of nonlinear partial differential equations and include boundary conditions. From a physical point of view, the situation is simpler. The Navier–Stokes equation describes the acceleration of a fluid subjected to interior forces. We add terms to the equation representing the external pressure and viscous forces at the boundary. It then describes the acceleration in a system exposed to interior and external forces. A common procedure then is to integrate the equation. We have done just that.

The Navier–Stokes is recast in the form of a divergence that is zero, permitting immediate integration. By invoking symmetry properties, the resulting expression may be partitioned into two equations that may be solved separately. One equation gives a linear partial differential equation for the velocity potential, permitting the calculation of a rotational part of the velocity field. The other equation reduces to a quadratic algebraic equation for components of a velocity field. These two solutions may be added, giving in a closed form an expression only depending on externally applied pressure and the value of the velocity and the derivatives of the velocity at the boundary.

The flow of an incompressible fluid of velocity $\boldsymbol{u}(\boldsymbol{r}, t)$ is governed by the Navier–Stokes equation (1) and by the equation of continuity (2). These equations are most often formulated as

$$\rho\partial_t \boldsymbol{u} + \rho\,(\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u} \equiv \rho\partial_t\boldsymbol{u} + \rho\boldsymbol{\omega}\times\boldsymbol{u} + \tfrac{1}{2}\nabla\rho\boldsymbol{u}^2 = -\nabla p - \mu\nabla\times\boldsymbol{\omega}, \tag{1}$$

$$\rho\nabla\cdot\boldsymbol{u} = Q(\boldsymbol{r}', t')\delta(\boldsymbol{r}-\boldsymbol{r}')\delta(t-t'). \tag{2}$$

Here, $\boldsymbol{\omega} = \nabla\times\boldsymbol{u}$ is the vorticity, p the pressure, ρ the density, μ the viscosity and $Q(\boldsymbol{r}', t')$ the amount of fluid injected into sources or removed by sinks from the system at a specified point and time. If the vorticity is considered as a skew-symmetric tensor, $\omega^{ji} = \partial^j u^i - \partial^i u^j$, a conjugate variable is the rate of strain, σ, defined from $\sigma^{ji} = \partial^j u^i + \partial^i u^j$. The Navier–Stokes equation is the law of force where the left-hand side contains the acceleration and the right-hand side the forces acting in the system. The density and the viscosity will be treated as constants. It follows from equation (2) that the Helmholtz decomposition applies, the velocity $\boldsymbol{u}$ is uniquely defined from an rotational part $\boldsymbol{u}_{\mathrm{rot}}$ determined by a vector potential $\boldsymbol{A}$, and a potential part $\boldsymbol{u}_{\mathrm{pot}}$ determined by a scalar potential ϕ as

$$\boldsymbol{u} = \boldsymbol{u}_{\mathrm{rot}} + \boldsymbol{u}_{\mathrm{pot}} = \nabla\times\boldsymbol{A} + \nabla\phi. \tag{3}$$

Here, ϕ has to satisfy the Laplace equation $\Delta\phi = 0$ if there are no sources and sinks. The vorticity is determined from the vector potential as $\boldsymbol{\omega} = \nabla(\nabla\cdot\boldsymbol{A}) - \Delta\boldsymbol{A}$. It is possible to choose $\boldsymbol{A}$ such that $\nabla\cdot\boldsymbol{A} = 0$ (see [9], for a discussion on the uniqueness of the Helmholtz decomposition).

Equation (1) describes the flow in a free space, but contains no information about how to take into account boundary conditions. We will integrate equation (1) and therefore include on its right-hand side the forces acting on the fluid at the boundary. These forces include pressure applied at inlets and outlets and viscous forces at solid boundaries.

We rewrite equation (2) to describe fluid entering or leaving the system at points $\boldsymbol{r}_Q$ on the boundary

$$\rho\nabla\cdot\boldsymbol{u}(\boldsymbol{r},t) = 2\Delta(\boldsymbol{r})\iint \mathrm{d}\boldsymbol{S}_Q\cdot\boldsymbol{u}(\boldsymbol{r}_Q,t')\rho G(\boldsymbol{r}-\boldsymbol{r}_Q)\delta(t-t'). \tag{4}$$

The integration is over all surfaces Q of inlets and outlets where the entrance, or exit, flow velocity is $\boldsymbol{u}(\boldsymbol{r}_Q)$. Here, G is the kernel, or Green's function, $G(\boldsymbol{r}) = (4\pi r)^{-1}$. As we are considering an incompressible fluid, the system responds instantaneously to any dynamical change of external pressure or the flow from sources. This justifies the use of a stationary Green's function in equation (4) satisfying the Laplace equation $\Delta G(\boldsymbol{r}) = -\delta(\boldsymbol{r})$. This would give a minus sign to the right-hand side of equation (4). However, another minus sign enters because the topological direction of the surface by definition is chosen such that the surface element vector $\mathrm{d}\boldsymbol{S}_Q$ points away from the system. The factor of 2 enters because when positioning point sources from equation (2) on a boundary surface, only half the flux will enter the system, the other half 'on the other side of the wall' will contribute to the environment not included in the system. This circumstance may be explained in more detail by the following argument.

Suppose $F(x, y, z)$ is a function defined in a 3D space. Then, if one approaches the plane $z = 0$ in a direction normal to a point $(x, y, 0)$,

$$\lim_{z\to\pm 0}\partial_z\iint G(x-x', y-y', z)F(x', y')\,\mathrm{d}x'\,\mathrm{d}y' = \mp\frac{1}{2}F(x, y). \tag{5}$$

We rewrite equation (4) as

$$\rho\partial_j u^j(\boldsymbol{r},t) - \partial_j\partial^j 2\iint \mathrm{d}\boldsymbol{S}_Q\cdot\boldsymbol{u}(\boldsymbol{r}_Q,t')\rho G(\boldsymbol{r}-\boldsymbol{r}_Q)\delta(t-t') = 0. \tag{6}$$

Here, we have applied a convention, a summation is to be carried out over identical upper and lower indices. Next, we multiply equation (6) by the i-component of the velocity. Using simple derivation rules and applying the fact that the second term of equation (6) is zero except at inlets and outlets, one arrives at

$$\rho u^j\partial_j u^i = \rho\partial_j\left(u^j_j u^i\right) - \partial_j\partial^j 2\iint \mathrm{d}\boldsymbol{S}_Q\cdot\boldsymbol{u}(\boldsymbol{r}_Q,t')u^i(\boldsymbol{r}_Q,t')\rho G(\boldsymbol{r}-\boldsymbol{r}_Q)\delta(t-t'). \tag{7}$$

The left-hand side of equation (7) is identical to the second nonlinear term of equation (1). We continue by adding to the right-hand side of equation (1) the forces acting on the system at the boundary in the form of an externally applied pressure at inlets and outlets, and viscous stress acting at the boundary. Suppose we maintain at a point $\boldsymbol{r}_Q$, belonging to an inlet or outlet, a pressure $p(\boldsymbol{r}_Q, t')$ applied in the direction $-\boldsymbol{n}_Q$ if $\boldsymbol{n}_Q$ is directed from the fluid. One should then complement equation (1) on the right-hand side with a term $-\boldsymbol{n}_Q p(\boldsymbol{r}_Q, t')\delta(\boldsymbol{r}-\boldsymbol{r}_Q)\delta(t-t')$.

The force from viscous stress at the surface element $\mathrm{d}\boldsymbol{S}_B$ at a point $\boldsymbol{r}_B$ on the boundary is given by the term $\mu\,\mathrm{d}\boldsymbol{S}_B\times[\nabla\times\boldsymbol{u}(\boldsymbol{r},t)\delta(\boldsymbol{r}-\boldsymbol{r}_B)] = \mu\nabla[\boldsymbol{u}(\boldsymbol{r},t)\delta(\boldsymbol{r}-\boldsymbol{r}_B)\cdot\mathrm{d}\boldsymbol{S}_B] - \mu[\mathrm{d}\boldsymbol{S}_B\cdot\nabla]\boldsymbol{u}(\boldsymbol{r},t)\delta(\boldsymbol{r}-\boldsymbol{r}_B) = \mu\,\mathrm{d}\boldsymbol{S}_B\cdot\nabla\boldsymbol{u}(\boldsymbol{r},t)\delta(\boldsymbol{r}-\boldsymbol{r}_B)$, an expression defining the dynamic viscosity μ. The last equality is obtained if the boundary conditions are satisfied. Writing these two contributions to the right-hand side of equation (1) with the help of Green's functions, one arrives at the contribution

$$\partial_j\left[\partial^j 2\iint G(\boldsymbol{r}-\boldsymbol{r}_Q)p(\boldsymbol{r}_Q,t)\,\mathrm{d}\boldsymbol{S}_Q - \mu\partial^j 2\iint G(\boldsymbol{r}-\boldsymbol{r}_B)\,\mathrm{d}\boldsymbol{S}_B\cdot\nabla\boldsymbol{u}(\boldsymbol{r}=\boldsymbol{r}_B,t)\right], \tag{8}$$

where the surface integrals are to be performed over the interior of the system boundary. The factor of 2 in this equation appears for a reason similar to the factor of 2 in equation (4). One may now collect terms from equations (7) and (8) by defining the tensors P, S and Ω

$$P^{ji} = -2\partial^j \iint p(\boldsymbol{r}_p, t) G(\boldsymbol{r} - \boldsymbol{r}_p)\, \mathrm{d}S_p^i, \tag{9}$$

$$R^{ji} = \partial^j \iint G(\boldsymbol{r} - \boldsymbol{r}_B)\, \mathrm{d}\boldsymbol{S}_B \cdot (\mu\nabla - \rho\boldsymbol{u}) u^i(\boldsymbol{r}_B, t), \tag{10}$$

$$\Omega^{ji} = R^{ji} - R^{ij} = \epsilon^{ijk}\Omega_k \qquad \text{and} \qquad S^{ji} = R^{ji} + R^{ij}. \tag{11}$$

The right-hand side of equation (9) is obtained from the first term inside the bracket of equation (8) after a partial integration and using the fact that the pressure at the rim of an opening is finite for viscous flow. The tensor P is diagonal if the pressure gradient is normal to the system boundary at inlets and outlets. We will assume this is the case. The tensor Ω is skew-symmetric and is related to the amount of vorticity transferred to the system, or generated, at boundaries. The first relation of equation (11) defines the vector $\boldsymbol{\Omega}$. In the definition of these tensors, the area of inlets and outlets has been assumed situated at the system boundary. The trace, the sum of the diagonal elements, of the tensors S and Ω, $\mathrm{tr}(S)$ and $\mathrm{tr}(\Omega)$, are zero.

Collecting all terms from equations (3), (7) and (8), equation (1) may be rewritten for the i-component as

$$\partial_j[\rho\partial_t\phi\delta^{ij} + \rho\partial_t\epsilon^{ijk}A_k + \rho u^i u^j + p\delta^{ij} + \mu\epsilon^{ijk}\omega_k + (S^{ji} + P^{ji} + \Omega^{ji})\delta(t - t')] = 0, \tag{12}$$

where ϵ^{ijk} is the completely skew-symmetric Levi-Civita tensor. Equation (12) has the form of a divergence being zero. Now, as stated after equation (2), the disappearing of the divergence leads to the existence of potentials. Applying this result to equation (12) leads to an equation for a second-rank tensor. The expression inside the brackets of equation (12) must be equal to a second-rank tensor T satisfying $\partial_j T^{ij} = \partial_i T^{ij} = 0$. That equation may be split into its symmetric and skew-symmetric parts yielding the two equations

$$\rho\partial_t\phi\delta^{ij} + [\rho u^i u^j + p\delta^{ji} + P^{ij} + S^{ij}]\delta(t - t') = T^{ij}, \tag{13}$$

$$\epsilon^{ijk}\rho\partial_t A_k - \mu\epsilon^{ijk}\Delta A_k + \Omega^{ji}\delta(t - t') = \epsilon^{ijk}B_k, \tag{14}$$

where T is an arbitrary symmetric tensor and $\boldsymbol{B}$ an arbitrary irrotational vector. T and $\boldsymbol{B}$ appear in analogy with a constant of integration in a one-dimensional analysis. If they cannot be given a physical interpretation, they may be put equal to zero. Lowering one index and contracting equation (13) gives the relation

$$\partial_t\rho\phi D + \rho\boldsymbol{u}^2 + pD + \mathrm{tr}(P) = 0, \tag{15}$$

where D is the space dimension of the system. This equation is employed to calculate the pressure in the flow provided the velocity and the tensors are known. It remains to treat the case of the non-diagonal elements of equation (13), that is the case $i \neq j$, and equation (14).

The off-diagonal part of equation (13) now reads

$$\rho u^i u^j + S^{ij} = 0, \qquad i \neq j, \tag{16}$$

and has the solution $\boldsymbol{U}$ with components

$$U^i(\boldsymbol{r}, t) = \pm\sqrt{\frac{S^{ij}S^{ik}}{\rho S^{jk}}}, \qquad i \neq j \neq k. \tag{17}$$

This velocity is, in general, not irrotational.

Equation (14) is a linear differential equation for the vector potential $\boldsymbol{A}$. To solve this equation, employ the integral kernel of the diffusion equation, $K(\boldsymbol{r}, t)$, given in the form

$$(\mu\Delta - \nu^{-1}\partial_t)K(\boldsymbol{r}, t) = -\delta(\boldsymbol{r})\delta(\nu t) \quad \Rightarrow \quad K(\boldsymbol{r}, t) = \frac{\exp\left(-\frac{r^2}{4\nu t}\right)}{(4\pi\nu t)^{\frac{D}{2}}}, \tag{18}$$

where D is the space dimension and $\nu = \mu/\rho$ is the kinematic viscosity. Multiplied by a unit vector, K defines a solenoidal vector potential. Vortex models derived from this kind of potential have been shown to account for the decay of vortex singlets generated in a tank [10]. Models of vortex doublets derived from this kind of vector potential reproduce the energy spectrum of 2D turbulent flows generated in the laboratory [11].

The solution of equation (14) then reads

$$\begin{aligned}\boldsymbol{A}(\boldsymbol{r}, t) &= \mu^{-1}\int_0^t \nu\, \mathrm{d}t' \iiint \mathrm{d}^3x' K(\boldsymbol{r} - \boldsymbol{r}', t - t')\boldsymbol{\Omega}(\boldsymbol{r}', t') \\ &= \nabla \times \int_0^t \mathrm{d}t' \iint \Gamma(\boldsymbol{r} - \boldsymbol{r}_B, t - t')\, \mathrm{d}\boldsymbol{S}_B \cdot (\nu\nabla - \boldsymbol{u}(\boldsymbol{r}_B))\boldsymbol{u}(\boldsymbol{r} = \boldsymbol{r}_B, t').\end{aligned} \tag{19}$$

The second equality is obtained after a partial integration, applying Stokes' theorem to an integral over a closed surface, and an integration over all space. The gradient in the parenthesis of the integrand of equation (19) describes the vorticity generated at the system boundary, solid boundaries as well as inlets and outlets. The velocity in the parenthesis of the integrand of equation (19) describes the vorticity entering the system with bulk flow at inlets and outlets.

The function Γ is a solution to $\Delta(\boldsymbol{r})\Gamma(\boldsymbol{r} - \boldsymbol{r}_B, t) = -K(\boldsymbol{r} - \boldsymbol{r}_B, t)$, and is given by $\Gamma(\boldsymbol{r}, t) = G(r)\mathrm{erf}\left(\frac{r}{\sqrt{4\nu t}}\right)$, where erf is the error function.

The rotational contribution to the internal velocity field from vorticity created at the boundary, $\boldsymbol{V}$, may now be determined by taking the rotation of the vector potential of equation (19)

$$\boldsymbol{V}(\boldsymbol{r}, t) = \int_0^t \mathrm{d}t' \iint K(\boldsymbol{r} - \boldsymbol{r}_B, t - t')\, \mathrm{d}\boldsymbol{S}_B \cdot (\nu\nabla - \boldsymbol{u}(\boldsymbol{r}_B))\, \boldsymbol{u}(\boldsymbol{r}_B, t'). \tag{20}$$

This velocity field defines a rotational part of the velocity in terms of the externally applied pressure and the value of the velocity and derivatives of the velocity at boundary points.

To compute the velocity field from equations (17) and (20), one needs information about the velocity and velocity gradients everywhere at the boundary. One way to obtain this information is to find a solution to the equation of continuity, equation (2) satisfying the no-slip boundary conditions, the velocity at the boundary is zero. One may proceed as follows. The bulk flow between inlets and outlets in an infinite space is given by the velocity field

$$\boldsymbol{v}_Q(\boldsymbol{r}, t) = 2\iint \mathrm{d}\boldsymbol{S}_Q \cdot \nabla(\boldsymbol{r})G(\boldsymbol{r} - \boldsymbol{r}_Q)\boldsymbol{v}(\boldsymbol{r}_Q, t). \tag{21}$$

Here, we have employed a relation of the following kind. On the boundary, the velocity normal to the surface, $\boldsymbol{v}_n$, and the tangential velocity, $\boldsymbol{v}_t$, are given by $\boldsymbol{v}_n(\boldsymbol{r}_B, t) = \boldsymbol{n}_B(\boldsymbol{n}_B \cdot \boldsymbol{v}_Q(\boldsymbol{r}_B, t))$ and $\boldsymbol{v}_t(\boldsymbol{r}_B, t) = \boldsymbol{v}(\boldsymbol{r}_B, t) - \boldsymbol{n}_B(\boldsymbol{n}_B \cdot \boldsymbol{v}_Q(\boldsymbol{r}_B, t))$, respectively. Here, $\boldsymbol{n}_B$ is a unit vector normal to the boundary surface at $\boldsymbol{r}_B$ oriented in a direction from the fluid. The appearance of the factor of 2 has been explained above in the paragraph following equation (4). One may now modify equation (21) to get a velocity field satisfying the boundary conditions by subtracting the normal and tangential velocities by using the integral kernels G and K,

$$\begin{aligned}\boldsymbol{v}(\boldsymbol{r}, t) = \boldsymbol{v}_Q(\boldsymbol{r}, t) &- 2\iint \mathrm{d}\boldsymbol{S}_B \cdot \nabla(\boldsymbol{r})G(\boldsymbol{r} - \boldsymbol{r}_B)\boldsymbol{v}_n(\boldsymbol{r}_B, t) \\ &- 2\int_0^t \nu\, \mathrm{d}t' \iint \mathrm{d}\boldsymbol{S}_B \cdot \nabla(\boldsymbol{r})K(\boldsymbol{r} - \boldsymbol{r}_B, t - t')\boldsymbol{v}_t(\boldsymbol{r}_B, t').\end{aligned} \tag{22}$$

The Green's stationary function G has been employed instead of the kernel K in the term treating the normal velocity component for the same reasons as those stated after equation (4) above. The fact that the normal component is zero is a consequence of the equation of continuity. As we are treating the case of an incompressible fluid, there is no time dependence in this equation.

The velocity field $\boldsymbol{v}$ here describes the bulk flow and the rotational part of the velocity field created at boundaries. One may note that the last term of equation (22) is twice the expression of equation (20) if one identifies the velocities of the integrands. A factor of 2 is due to the split of the tensor R into symmetric and skew-symmetric tensors, Ω and S. The expression of equation (20) may now be determined from the last term of equation (22) that is, from information based on the equation of continuity and the location of boundaries, inlets and outlets.

The part of the velocity field that may still be lacking is the one stemming from vortex doublet formation, two counter-rotating vortex singlets, as observed in pipe flow subjected to an external disturbance [12] or in vortex rings just before transition towards turbulence [13]. No net vorticity can be created in the interior, but nothing prevents the formation of vortex doublets in a shearing flow. This may be coupled to another feature of the Navier–Stokes equation.

It is a partial differential equation of the parabolic type, containing only single derivative terms with respect to time. Information then travels with an infinite speed. To describe dynamical features caused by local pressure fluctuations, the flow equations should be complemented by terms containing second derivatives with respect to time, in effect transforming them to wave equations. It would not change any of the above results obtained prior to equation (14). This equation would be complemented by a term representing a second time derivative of the vector potential, a term $\mu c^{-2}\partial_t^2$ may be added to the left hand side of equation (14). Here, c is a reference propagation velocity, for example the velocity of sound in a medium. It will result in an integral kernel not just decaying exponentially with time, but also being capable of reproducing an oscillatory behaviour. The Green's function in this case may be written as

$$G(x) = -\frac{1}{(2\pi)^4}\int_{C_R} \mathrm{d}^3k\,\mathrm{d}\omega\,\frac{\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}}{\boldsymbol{k}^2-\omega^2/c^2-\mathrm{i}\omega/\nu}, \qquad (23)$$

where the integration path in the ω-plane is along the real axis and both poles are in the lower half-plane if $\omega > 0$. This Green's function may be recast using Bessel functions [14]. Equation (20) is still valid if one replaces the integral kernal K by the Green's function of equation (23).

It has been demonstrated that the flow equations by integration may be reduced to one linear partial differential equation and one algebraic equation quadratic in the velocity. This permits the computation of the velocity field and the pressure in the flow in terms of the externally applied pressure and the value of the velocity and velocity gradients at boundary points.

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